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Linear Systems and Normality

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This paper is concerned with responses of linear systems to non-Gaussian random excitation and with the measurement of the departure of the responses from Gaussian behavior. First, we show the classical Rosenblatt result and its nonapplicability to the most popular practical systems described by differential equations of first and second order. Then, using a simple measure of departure from normality (the asymmetry and excess coefficients) and performing numerical calculations, we give quantitative information about the effect of system parameters and the radius of correlation of the excitation process on the distance from normality.

KEY WORDS: Linear systems; random excitation; Gaussian processes; limit theorems; departure from normality.

1. INTRODUCTION

One of the most common assumptions in the theoretical analysis of random phenomena is the normality of the processes under consideration. For instance, the stochastic analysis of linear systems is greatly simplified if it can be assumed that the random processes treated are Gaussian. In the engineering theory of random signals it is usually believed that if a stationary stochastic process is the input to a linear system, then the response is "approximately normally distributed" as the bandwidth of the system tends to zero. Obviously, such a statement cannot be true in general and in the cases when it can be justified we need not only the asymptotic result but also an estimation of the departure of the response from normality in any practical situation.

Therefore, it is of great importance to establish some criteria for the Gaussian assumption in the stochastic analysis of linear systems. Even

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when general answers are impossible it seems to be very useful to analyze the problem of normality and nonnormality, taking particular problems of practical interest.

Let the input to a linear system be denoted by $X(t)$. We shall assume that $X(t)$ is a given stationary, non-Gaussian, stochastic process. The resulting response is represented as

$$Y(t) = \int_0^t p(t,\tau)X(\tau)\,d\tau \tag{1.1}$$

where $p(t,\tau)$ is a unit response function. The integral (1.1) can be represented as a finite sum of random variables

$$Y(t) \approx \sum_{i=1}^{n} \zeta_i, \qquad \zeta_i = \int_{(j-1)\Delta}^{j\Delta} p(t,\tau)X(\tau)\,d\tau \tag{1.2}$$

where Δ denotes the length of an elementary interval in the division of the interval $[0,t]$ into n parts. If the variables ζ_i were independent then—by virture of the classical central limit theorem—the distribution of $Y(t)$ could be considered approximately Gaussian as $n \to \infty$. But the variables ζ_i are dependent, being functions of the same stochastic process. The central limit theorem has been extended to the case when random variables are dependent (cf. Refs. 1, 2). This more general form of the central limit theorem imposes, however, new conditions on the random variables under consideration. It was shown (cf. Refs. 2, 3) that the process $X(t)$ in (1.1) must satisfy a strong mixing condition. In addition to this requirement, the system characteristic $p(t,\tau)$—as was shown in Rosenblatt's classical paper[4]—has to satisfy appropriate analytical conditions.

The papers mentioned above constitute a valuable contribution to the understanding of the mathematical mechanism of the occurrence of Gaussian responses in linear systems, but they provide only qualitative and asymptotic results. In specific situations one is faced with the problem of testing the mixing condition and other assumptions. Furthermore, the problem of appropriate bounds for the departure of the response $Y(t)$ from normality remains. From the point of view of applications it is of importance to establish some quantitative results. A significant step in this direction has been made in Refs. 5–7.

In Refs. 5 and 6 the authors consider the shot noise process, that is, a stochastic process $Y(t)$ defined by

$$Y(t) = \sum_i p(t,t_i) \tag{1.3}$$

where t_i are the random times of a Poisson process with average intensity $\kappa(t)$. The process (1.3) can be considered as the output of a linear system with unit response function $p(t, \tau)$ subjected to a sequence of impulses at the times t_i.

With $F_t(y)$ the distribution function of $Y(t)$ and $\Phi_t(y)$ that of a Gaussian process with the same mean and variance as $Y(t)$, it has been shown in Ref. 6 that for any t

$$|F_t(y) - \Phi_t(y)| \leqslant \frac{4}{3}\left[\frac{2\pi I_3^2(t)}{I_2^3(t)}\right]^{1/2} \tag{1.4}$$

where

$$I_n(t) = \int_{-\infty}^{+\infty} \kappa(\tau)|p(t,\tau)|^n \, d\tau \tag{1.5}$$

If, additionally, $\kappa(t) = \kappa = \text{const}$ and $p(t,\tau) = p(t-\tau)$ and the function $p(\theta)$ is band-limited by ω_c, then

$$|F_t(y) - \Phi_t(y)| \leqslant 2(\omega_c/\kappa)^{1/2} \tag{1.6}$$

Hence, if $\omega_c/\kappa \to 0$, then $F_t(y)$ tends to a normal distribution.

Reference 7 constitutes an extension of Ref. 6 to a wider class of excitation of linear systems. The author investigates the normality of the response $Y(t)$ of a narrow-band system when the input $X(t)$ is a stationary process with finite time of dependence [if a is a finite number, then the random variables $X(t)$ and $X(t+u)$ are independent for $u > a$]. Using the classical Berry–Essen inequality, the author establishes an upper bound for the departure of the response from normality. This bound is expressed in terms of the bandwidth of the system and spectral density and the third absolute moment of the excitation. It should be noted that a systematic and uniform approach to measuring the distance from normality with applications to discrete linear filters is presented in Ref. 8.

Though the results described above provide an essential contribution to the Gaussian analysis of linear systems, the situation is still far from being clear—especially when one is concerned with *real* physical systems. The purpose of the present paper is to give further (and more concrete) information concerning responses of linear systems to non-Gaussian excitation. First, we show the classical Rosenblatt result and its nonapplicability to the most popular practical systems described by differential equations of first and second order. Then, using a simple measure of departure from normality (the asymmetry and excess coefficients) and performing numerical calculations, we provide the answer to the question: what are the values of system parameters and the radius of correlation of the excitation process for which the departure from normality of the response is less or greater than a given quantity? The paper is based on unpublished notes[9] and a thesis.[10]

2. ROSENBLATT'S THEOREM AND REAL SYSTEMS

Let us consider a linear system (filter) defined by relation (1.1). Let

$$P(t) = \int_0^t p^2(t,\tau)\,d\tau \tag{2.1}$$

In Ref. 4 Rosenblatt introduces the following assumptions.

Assumptions A:

1. $P(t) \to \infty$ as $t \to \infty$.
2. The function $p_t(\tau) = p(t,\tau)$ increases slowly, so that (a)

$$\int_{A(t)} |p_t(\tau)|^2 d\tau = o(P(t)) \qquad \text{as} \quad t \to \infty$$

for every sequence of subsets $A(t) \subset (0,t)$ whose Lebesgue measure $m(A(t))$ is such that $m(A(t)) = o(t)$ uniformly with respect to $(1/t)m(A(t))$ as $t \to \infty$; and, (b) $p(t,\tau) = O([P(t)]^{1/2})$ uniformly in t as $t \to \infty$.

3. We have the following:

$$\lim_{t\to\infty} \frac{1}{P(t)} \int_0^{t-|h|} p(t,\tau+|h|)p(t,\tau)\,d\tau = \rho(h)$$

which exists for every h and is continuous in h.

The limit function $\rho(h)$ is a nonnegative-definite function and by virtue of the Bochner theorem it has a representation

$$\rho(h) = \int_{-\infty}^{+\infty} e^{ih\lambda}\,dM(\lambda)$$

where $M(\lambda)$ is a nondecreasing, bounded function.

Assumptions B:

1. The process $X(t)$ satisfies the strong mixing condition, which can be formulated as follows (cf. Ref. 2): let $\mathcal{B}_t$ be a Borel field of events generated by random variables $X(u)$, $u \leqslant t$, and $\mathcal{F}_\tau$ be a Borel field of events generated by random variables $X(u)$, $u \geqslant \tau$; there exists a nonnegative function $\varphi(s)$ defined on $[0,\infty)$ and $\varphi(s) \to 0$ as $s \to \infty$ such that for any pair of events $B \in \mathcal{B}_t$ and $F \in \mathcal{F}_\tau$

$$|\mathcal{P}(B \cap F) - \mathcal{P}(B)\mathcal{P}(F)| < \varphi(\tau - t)$$

where $\mathcal{P}(B)$ denotes a probability of event B.

2. Let $E[X(t)] = 0$. The moments

$$E\big[X(t_1)X(t_2)\big] = K(t_2 - t_1)$$

$$E\big[X(t_1)X(t_2)X(t_3)X(t_4)\big] = R(t_2 - t_1, t_3 - t_1, t_4 - t_1)$$

are absolutely integrable over one- and three-dimensional space, respectively, and the spectral density $g(\lambda)$ of the process $X(t)$ is positive everywhere.

Rosenblatt has shown that under the above assumptions

$$\frac{1}{[P(t)]^{1/2}} \int_0^t p(t,\tau) X(\tau)\, d\tau \tag{2.2}$$

is asymptotically normally distributed with mean zero and variance equal to

$$2\pi \int_{-\infty}^{+\infty} g(\lambda)\, dM(\lambda)$$

The conditions of the theorem are not easy to verify directly. In particular, the problem of testing the strong mixing condition in real physical situations is rather involved (cf. Ref. 11). Only for Gaussian stationary processes can this condition be replaced by a simpler one,[(12)] but we are interested here in the non-Gaussian case.

Let us consider dynamical systems described by the differential equations

$$\text{(I)} \qquad \frac{dY}{dt} + aY = X(t), \qquad a > 0$$

$$\text{(II)} \qquad \frac{d^2Y}{dt^2} + 2h\frac{dY}{dt} + \omega_0^2 Y = X(t)$$

where a, h, and ω_0 are positive constants. The last equation is widely used in the analysis of various physical phenomena and constitutes a basic model in vibration theory; usually h characterizes the damping in the system and ω_0 is a natural frequency. The unit response functions are, respectively,

$$p(t,\tau) = \exp[-a(t-\tau)] \tag{2.3}$$

$$p(t,\tau) = \frac{1}{\omega_h} \sin \omega_h (t-\tau) \exp[-h(t-\tau)] \tag{2.4}$$

where we have assumed that $\omega_h^2 = \omega_0^2 - h^2$ is positive.

Leaving out the conditions B concerning the excitation process, one easily sees that in the case of system (I)

$$P(t) = \int_0^t p^2(t,\tau)\, d\tau = \frac{1}{2a}(1 - e^{-2at})$$

and condition A.1 is not satisfied. Similarly, in case (II)

$$P(t) = \int_0^t p^2(t,\tau)\, d\tau = \frac{\omega_h^3}{4\omega_0^2 h} - \frac{\omega_h}{8\omega_0^2} e^{-2ht}\left[4h \sin^2 \omega_h t + 2\omega_h \sin 2\omega_h t + \frac{2\omega_h^2}{h}\right]$$

and it is seen that condition A.1 is also not satisfied here. Therefore, the vibration analysis of damped linear systems subjected to non-Gaussian excitation cannot use Rosenblatt's result. For finite values of system parameters a, h, ω_0 the band of the systems considered is not infinitely narrow and the response departs from normality significantly. In the next section we shall show how, in the realistic cases to be considered, departure from normality is affected by the correlation parameter of the excitation and the bandwidth of the system.

3. REAL LINEAR SYSTEMS AND NORMALITY

In order to obtain quantitative results a tractable, specific form of non-Gaussian excitation is taken into consideration, namely,

$$X(t) = Z^2(t) \tag{3.1}$$

where $Z(t)$ is a stationary and Gaussian process such that its average is equal to zero and the correlation function is $K_Z(\tau) = \sigma_Z^2 e^{-\alpha|\tau|}$, $\alpha > 0$. In this case

$$m_X = E[X(t)] = \sigma_Z^2, \qquad K_X(t) = \sigma_X^2 e^{-\beta|\tau|}, \qquad \sigma_X^2 = 2\sigma_Z^4, \qquad \beta = 2\alpha \tag{3.2}$$

The one-dimensional probability density function of $X(t)$ is

$$f_X(x) = \begin{cases} \dfrac{1}{\sigma_Z (2\pi)^{1/2}} \dfrac{1}{x^{1/2}} \exp\left(-\dfrac{x}{2\sigma_Z^2} \right), & x > 0 \\ 0, & x < 0 \end{cases} \tag{3.3}$$

It is easily seen that departure from normality of the excitation process $X(t)$ is rather significant regardless of the measure of departure. Here, departure from normality is characterized by the asymmetry and excess coefficients defined by the formulas

$$\gamma_1 = \mu_3/\sigma^3, \qquad \gamma_2 = \mu_4/\sigma^4 - 3 \tag{3.4}$$

where σ is a standard deviation of the process considered and μ_3, μ_4 are the third- and fourth-order central moments, respectively. For a Gaussian distribution $\gamma_1 = \gamma_2 = 0$. Values of γ_1 and γ_2 different from zero characterize the departure of a one-dimensional distribution from normality. It is worth noting that the coefficients (3.4) are, in the case of the distribution (3.3), equal to $\gamma_1 = 3.9$, $\gamma_2 = 15$.

3.1. First-Order Dynamical Systems

Let us consider first the system (I), where the excitation process is given by (3.1), (3.2). Equation (I) is equivalent to the following system of equations:

$$\begin{aligned} dY/dt &= -aY + Z^2(t) \\ dZ/dt &= -\alpha Z(t) + \sigma_Z (2\alpha)^{1/2} \xi(t) \end{aligned} \tag{3.5}$$

where $\xi(t)$ is Gaussian white noise with $E[\xi(t)] = 0$, $K_\xi(\tau) = \delta(\tau)$. The vector process $[Y(t), Z(t)]$ constitutes a diffusion Markov process. The Fokker–Planck–Kolmogorov equation for the probability density function $f(y, z; t)$ has the following form:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial y}\left[(z^2 - ay)f\right] - \alpha \frac{\partial}{\partial z}(zf) - \alpha\sigma_Z^2 \frac{\partial^2 f}{\partial z^2} \tag{3.6}$$

Looking for the stationary solution of the above equation and making use of the definition of moments

$$m_k = \int_{-\infty}^{+\infty} y^k f_y(y)\,dy = \int_{-\infty}^{+\infty} y^k \int_{-\infty}^{+\infty} f(y,z)\,dz\,dy, \qquad k = 1,2,3,4$$

one obtains a recursive set of equations for the moments m_k. After appropriate transformations we get

$$m_1 = \frac{1}{a}\sigma_Z^2, \qquad m_2 = \frac{\sigma_Z^4}{a^2}\,\frac{3a + 2\alpha}{a + 2\alpha}, \qquad \sigma_Y = \frac{2\sigma_Z^2}{\left[2a(a + 2\alpha)\right]^{1/2}}$$

$$m_3 = \frac{1}{a(a + \alpha)(a + 4\alpha)}\left[15\sigma_Z^6 + (13a + 4\alpha)\alpha\sigma_Z^2 m_2\right]$$

$$m_4 = \frac{1}{a(a + 2\alpha)(a + 6\alpha)(3a + 2\alpha)}\Big\{\left[90a(a + \alpha) + 4(a + 6\alpha)(5a + \alpha)\right]$$

$$\times\,\alpha\sigma_Z^2 m_3 - 90\alpha^2\sigma_Z^4 m_2 + 315\sigma_Z^8\Big\}$$

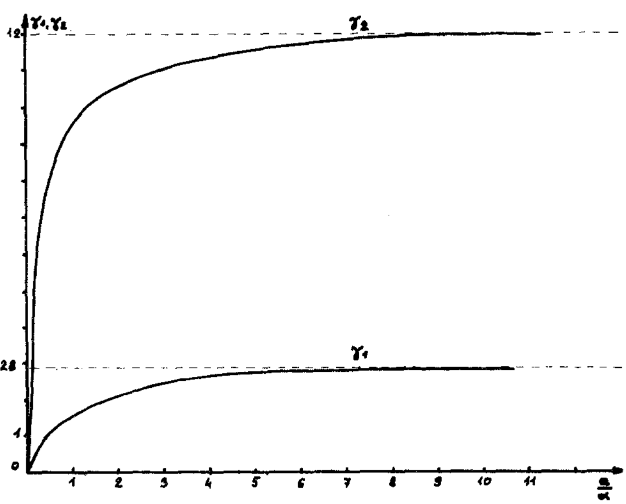

Fig. 1

Making use of the relations

$$\mu_3 = m_3 - 3m_1m_2 + 2m_1^3, \qquad \mu_4 = m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4$$

and the definition (3.4), one obtains the following result:

$$\gamma_1 = \frac{2}{a+\alpha}\left[2a(a+2\alpha)\right]^{1/2}$$

$$\gamma_2 = 3\left[\frac{15a^2 + 25a\alpha + 2\alpha^2}{(a+\alpha)(3a+2\alpha)} - 1\right] \tag{3.7}$$

It is seen that for $\alpha = \text{const}$, then $\gamma_1 \to 0$, $\gamma_2 \to 0$ as $a \to 0$ (the band of the filter is infinitely narrow). If a or a/α is finite, the coefficients γ_1 and γ_2 take significant values; this means that for infinite operating time [the values γ_1 and γ_2 correspond to a stationary solution of Eq. (3.6)] the response of a system differs significantly from a Gaussian one. Figure 1 shows the dependence of γ_1 and γ_2 on the ratio a/α.

3.2. Second-Order Systems

Let us consider now the vibratory linear system (II) with the process (3.1) as an excitation. In order to obtain the asymmetry and excess coefficients we shall calculate the following integrals:

$$m_1(t) = \int_0^t p(t,t_1)E[X(t_1)]\,dt_1 \tag{3.8}$$

$$m_2(t) = \int_0^t\int_0^t p(t,t_1)p(t,t_2)E[X(t_1)X(t_2)]\,dt_1\,dt_2 \tag{3.9}$$

$$m_3(t) = \int_0^t\int_0^t\int_0^t p(t,t_1)p(t,t_2)p(t,t_3) \times E[X(t_1)X(t_2)X(t_3)]\,dt_1\,dt_2\,dt_3 \tag{3.10}$$

$$m_4(t) = \int_0^t\int_0^t\int_0^t\int_0^t p(t,t_1)p(t,t_2)p(t,t_3)p(t,t_4) \times E[X(t_1)X(t_2)X(t_3)X(t_4)] \times dt_1\,dt_2\,dt_3\,dt_4 \tag{3.11}$$

where $p(t,t_1)$ is given by (2.4). The third and fourth moments occurring in (3.10) and (3.11) by virtue of the normality of $Z(t)$ can be expressed in terms of a given correlation function (3.2). Looking for the stationary response of system (II), one obtains

$$m_1 = \sigma_Z^2 C_1, \qquad m_2 = \sigma_Z^4 (C_1^2 + 2C_2)$$
$$m_3 = \sigma_Z^6 (C_1^3 + 6C_1C_2 + 8C_3) \tag{3.12}$$
$$m_4 = \sigma_Z^8 (C_1^4 + 12C_2C_1^2 + 12C_2^2 + 32C_1C_3 + 48C_4)$$

and

$$\gamma_1(Y) = 2^{3/2} C_3/C_2^{3/2}, \quad \gamma_2(Y) = 12C_4/C_2^2 \tag{3.13}$$

where

$$C_1 = \int_0^\infty p(\tau)\, d\tau$$
$$C_2 = \int_0^\infty \int_0^\infty p(\tau_1)p(\tau_2) \exp(-2\alpha|\tau_1 - \tau_2|)\, d\tau_1 d\tau_2$$
$$C_3 = \int_0^\infty \int_0^\infty \int_0^\infty p(\tau_1)p(\tau_2)p(\tau_3) \times \exp(-\alpha|\tau_1 - \tau_2| - \alpha|\tau_3 - \tau_1| - \alpha|\tau_3 - \tau_2|)\, d\tau_1\, d\tau_2\, d\tau_3 \tag{3.14}$$
$$C_4 = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty p(\tau_1)p(\tau_2)p(\tau_3)p(\tau_4) \times \exp(-\alpha|\tau_1 - \tau_3| - \alpha|\tau_1 - \tau_4| - \alpha|\tau_2 - \tau_3| - \alpha|\tau_2 - \tau_4|)\, d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4$$

The constant quantities C_1, C_2, C_3, C_4 depend on the system parameters h, ω_h, and the correlation parameter α. The integrals (3.14) have been evaluated analytically.[10] The corresponding expressions are rather involved and they will not be presented here. The expressions for C_1, C_2, C_3, C_4 show, however, that these constants depend only on two parameters:

$$H = h/\alpha, \qquad \Omega = \omega_h/\alpha \tag{3.15}$$

In order to obtain quantitative results for $\gamma_1(Y)$ and $\gamma_2(Y)$ numerical calculations have been performed and the results are presented graphically in Figs. 2–5.

Figures 2a and 2b show the dependence of $\gamma_1(Y)$ and $\gamma_2(Y)$ on the parameter H when Ω is constant; it is seen that γ_1 and γ_2 tend to zero as $H \to 0$. Figures 3a–3d and 4a–4b illustrate the effect of damping coefficient h on γ_1 and γ_2; when $h \to 0$ the response tends to the Gaussian one (in this case the assumptions of Rosenblatt's theorem are satisfied). Figure 5 shows the influence of the correlation parameter α on γ_1 and γ_2.

It should be noticed that in both cases [systems (I) and (II)] linear

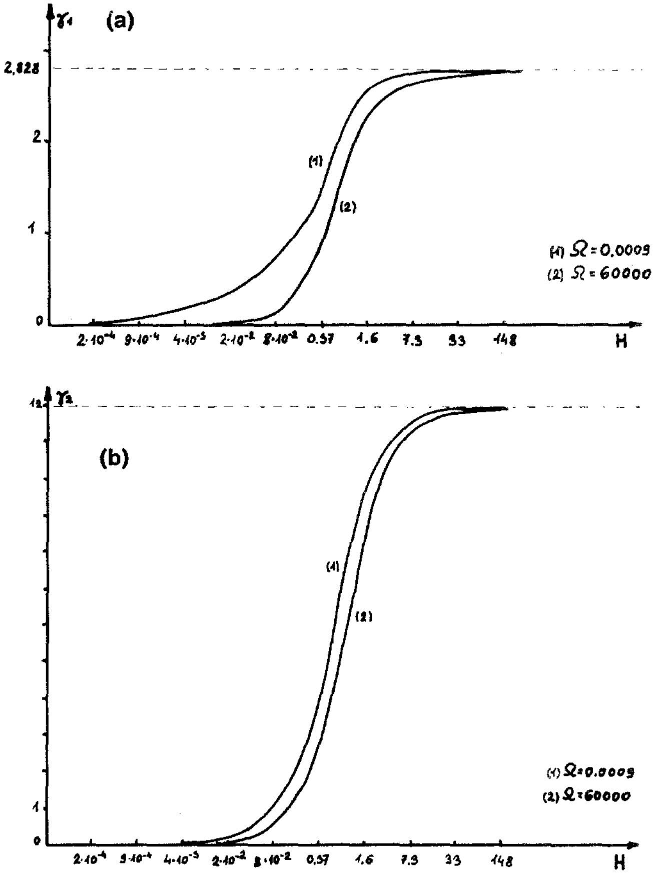

Fig. 2

filters "improve normality" in the sense that $\gamma_1(Y) < \gamma_1(X)$, $\gamma_2(Y) < \gamma_2(X)$ for all values of the system parameters. Nevertheless, the results obtained show that even for infinite operating time the response differs significantly from the Gaussian one when the bandwidth of the systems considered is finite, that is, when one deals with real, dynamical systems. The results shown in Fig. 5 refer to the realistic case when the system parameters h and ω_0 take values obtained from experimental investigations of the suspension of road vehicles (cf. Ref. 13). This figure shows that for $\alpha = 0.2$–0.4 (which are realistic for the correlation parameter of road roughness) $\gamma_1(Y) \approx 2.5$, $\gamma_2(Y) \approx 11$. This observation suggests that in this situation it would be rather unreasonable to approximate the response of the system by the

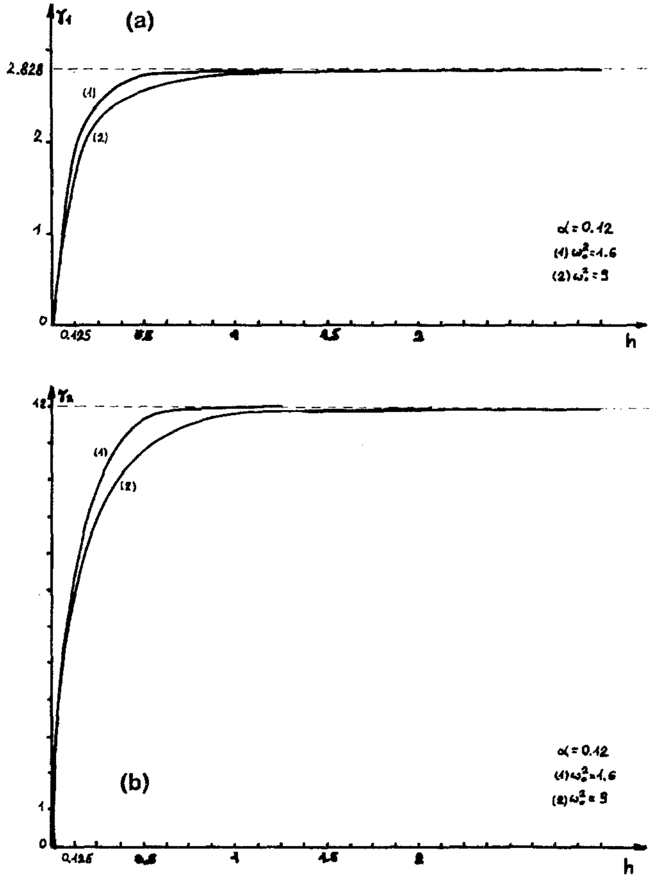

Fig. 3

Gaussian one. The curves presented in Figs. 2–5 can serve as pattern curves in the engineering analysis of vibratory systems subjected to random excitation.

3.3. Systems Subjected to Random Impulses

Let us consider briefly the response $Y(t)$ of a linear system with unit response function $p(t, \tau)$ to excitation in the form of a sequence of impulses occurring at random instants t_i of a Poisson process with intensity $\kappa(t)$. This response is of the form (1.3).

If the characteristic function of the process $Y(t)$ is denoted by $\Phi_t(\lambda)$,

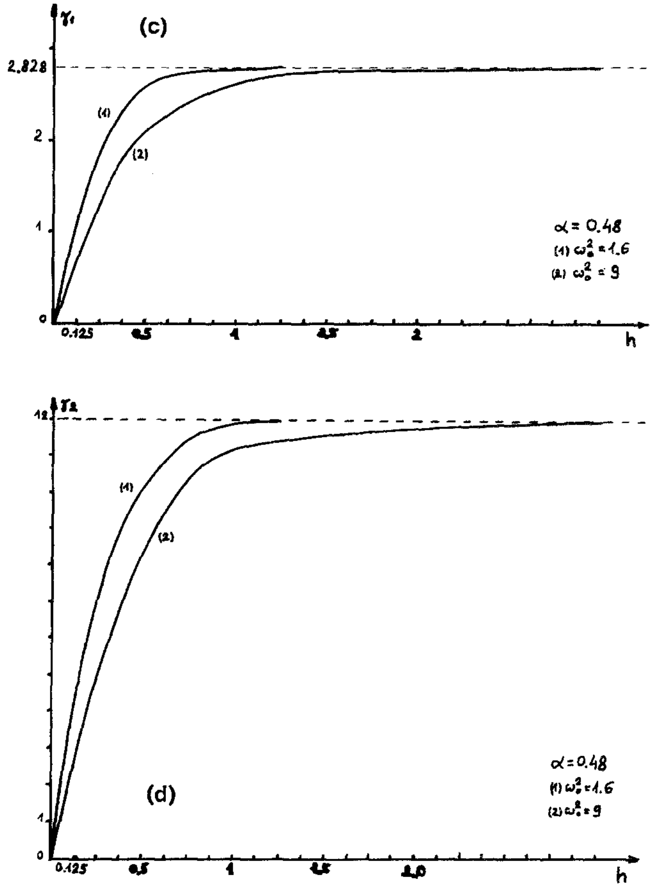

Fig. 3. Continued.

then (cf. Ref. 6)

$$\ln \Phi_t(\lambda) = \int_{-\infty}^{+\infty} \kappa(\tau)\left[e^{ip(t,\tau)\lambda} - 1 \right] d\tau \tag{3.16}$$

By use of the above formula one can easily obtain the first four moments of $Y(t)$ and consequently the coefficients γ_1 and γ_2; they are

$$\gamma_1(Y) = \left[\frac{I_3^2(t)}{I_2^3(t)} \right]^{1/2}, \qquad \gamma_2(Y) = \frac{I_4(t)}{I_2^2(t)} \tag{3.17}$$

where $I_n(t)$ is given by (1.5).

If $\kappa(t) = \kappa = \text{const}$, and in the case of stationary response of systems

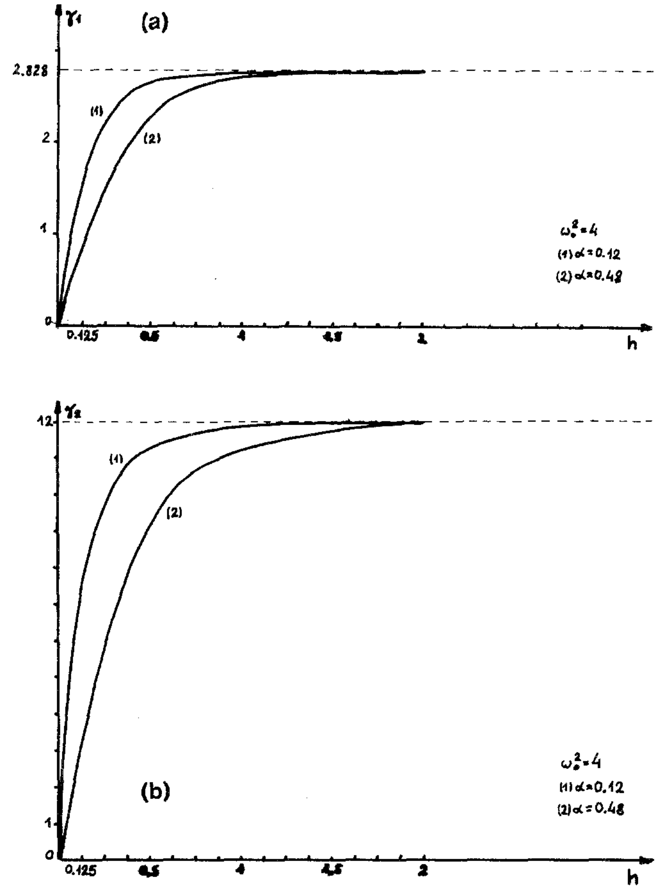

Fig. 4

(I) and (II), one obtains for system (I)

$$\gamma_1 = \frac{2^{3/2}}{3}\left(\frac{a}{\kappa}\right)^{1/2}, \qquad \gamma_2 = \frac{a}{\kappa} \tag{3.18}$$

and for system (II)

$$\gamma_1 = \frac{8}{3}\left[\frac{4h^3\omega_0^2}{\kappa\left(8h^2+\omega_0^2\right)^2}\right]^{1/2} \tag{3.19}$$

$$\gamma_2 = \frac{6h\omega_0^2}{\kappa\left(12h^2+\omega_0^2\right)} \tag{3.20}$$

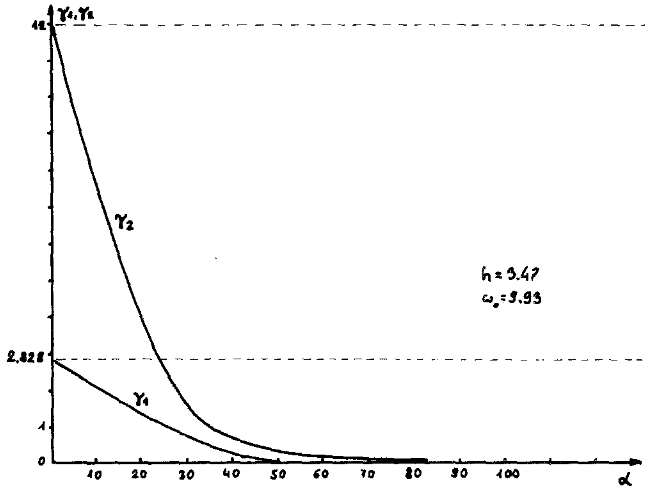

Fig. 5

When $a/\kappa \to 0$ or $h \to 0$, γ_1 and $\gamma_2 \to 0$ and the response $Y(t)$ (its one-dimensional distribution) can be considered as Gaussian. It is worth noting that for the values of h and ω_0 taken from the experimental examination of a road vehicle suspension,[(13)] that is, for $h = 2\pi \times 1.58 \times 0.35 = 3.47$ and $\omega_0 = 2\pi \times 1.58 = 9.93$, we have $\gamma_1 \approx 1.76\kappa^{-1/2}$, $\gamma_2 \approx 8.45\kappa^{-1}$.

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